J. Fluid Mech. (1973), *vol.* 58, *part* 1, *pp.* 161–164
Printed in Great Britain

A note on the application of a radiation condition for a source in a rotating stratified fluid

By A. RAMACHANDRA RAO
Department of Applied Mathematics, Indian Institute of Science, Bangalore 560012, India

(Received 9 October 1972)

The motion due to an oscillatory point source in a rotating stratified fluid has been studied by Sarma & Naidu (1972) by using threefold Fourier transforms. The solution obtained by them in the hyperbolic case is wrong since they did not make use of any radiation condition, which is always necessary to get the correct solution. Whenever the motion is created by a source, the condition of radiation is that the sources must remain sources, not sinks of energy and no energy may be radiated from infinity into the prescribed singularities of the field. The purpose of the present note is to explain how Lighthill's (1960) radiation condition can be applied in the hyperbolic case to pick the correct solution. Further, the solution thus obtained is reiterated by an alternative procedure using Sommerfeld's (1964) radiation condition.

1. Statement of the problem

The governing equation for the disturbance pressure p for the flow induced by an oscillatory point source in an unbounded rotating stratified fluid in which density variation is taken into account in the inertia terms has been derived by Sarma & Naidu (1972). It is given by

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2}+N^2\right)\nabla^2 p+(4\Omega^2-N^2)\frac{\partial^2 p}{\partial z^2}+\beta\left(\frac{\partial^2}{\partial t^2}+4\Omega^2\right)\frac{\partial p}{\partial z}\right]$$
$$=-\rho_0' e^{-\beta z}\left(\frac{\partial^2}{\partial t^2}+N^2\right)\left(\frac{\partial^2}{\partial t^2}+4\Omega^2\right)q\, e^{i\omega t}\delta(x)\,\delta(y)\,\delta(z), \quad (1)$$

where ∇^2 is the three-dimensional Laplacian operator, N is the Brunt–Väisälä frequency, Ω is the angular velocity of rotation, ρ_0' is the characteristic density, β is the stratification parameter, $q\,e^{i\omega t}$ is the strength of the source, ω is the oscillation frequency of the source and $\delta(x)$ is the Dirac delta function. Equation (1) is hyperbolic in the space variables when the frequency of oscillation of the source lies between the Brunt–Väisälä frequency and twice the angular velocity of rotation, i.e. when $4\Omega^2 < \omega^2 < N^2$ or $N^2 < \omega^2 < 4\Omega^2$.

2. Solution by a threefold Fourier integral

The solution of (1) following Lighthill (1960) is given in the form of a threefold Fourier integral:

$$p = \frac{-i\rho_0' q(N^2-\omega^2)\, e^{i\omega t}}{8\pi^3\omega}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left[i(lx+my+nz)\right]}{G(l,m,n)}\,dl\,dm\,dn, \qquad (2)$$

where

$$G = (n^2 - i\beta n - s^2/\lambda^2), \quad s^2 = l^2+m^2, \quad \lambda^2 = (4\Omega^2-\omega^2)/(\omega^2-N^2) \quad (>0). \qquad (3)$$

The integral in (2) is evaluated by performing the n integration by the method of residues and the l and m integrations by changing the variables suitably. The contribution to the n integration comes from the zeros of G, which are given by

$$n_2, n_2' = \tfrac{1}{2}[i\beta \pm (4s^2/\lambda^2-\beta^2)^{\frac{1}{2}}]. \qquad (4)$$

Thus the integrand has two poles lying in the upper half-plane ($z > 0$) and none in the lower half-plane. Sarma & Naidu (1972) have evaluated the n integration by choosing a semicircular contour in the upper half-plane which includes both the poles, and this is seen to be incorrect. Therefore there is apparently a difficulty in choosing a proper contour for integration. The difficulty disappears when we realize that a radiation condition should be used somehow to get the correct solution. Lighthill's radiation condition can be used when the poles lie on the real axis. By putting $n = \frac{1}{2}i\beta + n'$ in (2) the poles of the integrand are made to lie on the real axis and (2) becomes

$$p = \frac{-i\rho_0' q}{8\pi^3\omega}(N^2-\omega^2)\exp\left[i\omega t - \tfrac{1}{2}\beta z\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[i(lx+my)\right]dl\,dm\int_{-\infty}^{\infty}\frac{e^{in'z}}{g}\,dn', \qquad (5)$$

where

$$g = (n'^2 - n_1^2), \quad n_1 = (s^2/\lambda^2 - \tfrac{1}{4}\beta^2)^{\frac{1}{2}}. \qquad (6)$$

Physically, the effect of the transformation is to take out a factor $e^{-\frac{1}{2}\beta z}$ proportional to the square root of the undisturbed density. This factor in the pressure amplitude allows for those amplitude variations required by conservation of energy of the wave as it moves into regions of different density. What remains after this factor is taken out describes the other changes due to wave propagation.

By applying Lighthill's radiation condition (giving ω a small negative imaginary part $-i\epsilon$) we see that n_1 is displaced into lower half-plane ($z < 0$) and $-n_1$ into upper half-plane. Therefore for a semicircular contour taken in the upper half-plane, only

$$n' = -n_1 + i\epsilon\left(\frac{\partial g}{\partial \omega}\Big/\frac{\partial g}{\partial n'}\right)$$

gives a contribution and the residue at this pole is to be calculated as $\epsilon \to 0$. After the n' integration has been completed and the variables changed according to

$$l = s\cos\phi, \quad m = s\sin\phi, \quad x = r\cos\theta, \quad y = r\sin\theta, \qquad (7)$$

equation (5) becomes

$$p = \frac{-\rho_0' q(N^2-\omega^2)}{8\pi^2\omega}\exp\left[i\omega t - \tfrac{1}{2}\beta z\right]\int_0^{\infty}\int_0^{2\pi}\frac{\exp\left[-iz(s^2/\lambda^2-\frac{1}{4}\beta^2)^{\frac{1}{2}}\right]}{(s^2/\lambda^2-\frac{1}{4}\beta^2)^{\frac{1}{2}}} \times\exp\left[irs\cos(\phi-\theta)\right]s\,ds\,d\phi,$$

$$= \frac{-\rho_0'(N^2-\omega^2)}{4\pi\omega}\exp\left[i\omega t - \tfrac{1}{2}\beta z\right]\int_0^{\infty}\frac{\exp\left[-iz(s^2/\lambda^2-\frac{1}{4}\beta^2)^{\frac{1}{2}}\right]}{(s^2/\lambda^2-\frac{1}{4}\beta^2)^{\frac{1}{2}}}J_0(rs)\,s\,ds. \qquad (8)$$

The integral in (8) is evaluated by using the results given in Erdélyi *et al.* (1954, vol. 2, p. 31) and the expressions for p are

$$p = \frac{K}{\pi}\frac{\exp[i\omega t]}{(z^2-\lambda^2r^2)^{\frac{1}{2}}}\exp[-\tfrac{1}{2}\beta\{z+(z^2-\lambda^2r^2)^{\frac{1}{2}}\}] \quad \text{for} \quad z > \lambda r, \tag{9}$$

$$= \frac{-iK}{\pi}\frac{\exp[i\omega t]}{(\lambda^2r^2-z^2)^{\frac{1}{2}}}\exp[-\tfrac{1}{2}\beta\{z+i(\lambda^2r^2-z^2)^{\frac{1}{2}}\}] \quad \text{for} \quad z < \lambda r, \tag{10}$$

where $$K = -i\rho_0' q(4\Omega^2-\omega^2)/4\omega.$$

Equation (10) represents waves with surfaces of constant phase $\lambda^2r^2-z^2 =$ constant, which are hyperbolas asymptotic to the boundary of this region $z < \lambda r$. Equation (9) represents a non-wavy disturbance with energy decaying exponentially in the region $z > \lambda r$ beyond that boundary.

3. Alternative method of solution

By writing (1) in cylindrical polar co-ordinates (r, θ, z) with axisymmetry and taking Fourier tranforms with respect z, we obtain

$$\frac{d^2\overline{P}}{dr^2}+\frac{1}{r}\frac{d\overline{P}}{dr}+\lambda^2\gamma^2\overline{P} = \frac{-2K}{\pi r}\delta(r), \tag{11}$$

where $$\overline{p}e^{i\omega t} = \overline{P} = e^{i\omega t}\int_{-\infty}^{\infty} p(r,z)\,e^{inz}\,dz \tag{12}$$

and $\gamma^2 = n^2+i\beta n$.

The solution of (11), using variation of parameters, is given by

$$\overline{P} = AJ_0(\lambda\gamma r)-KY_0(\lambda\gamma r). \tag{13}$$

The constant A remains indeterminate under the condition that the disturbance should tend to zero at infinity. To determine A, the condition at infinity is replaced by the Sommerfeld's radiation condition, namely

$$\lim_{r\to\infty}\{r(\partial\overline{P}/\partial r+i\lambda\gamma\overline{P})\} = 0. \tag{14}$$

For large r, using the asymptotic forms of J_0 and Y_0, equation (13) can be written as

$$\overline{P} \sim C\left\{\frac{A+Ki}{\sqrt{r}}e^{i\lambda\gamma r}+\frac{A-Ki}{\sqrt{r}}e^{-i\lambda\gamma r}\right\}, \tag{15}$$

where C is a constant. The form (15) satisfies the radiation condition (14) if

$$A = -Ki. \tag{16}$$

If this value of A is used in (13), the inverse Fourier transform of $\overline{p}$ is given by

$$p = \frac{-Ki\,e^{i\omega t}}{2\pi}\int_{-\infty}^{\infty}[J_0(\lambda\gamma r)-iY_0(\lambda\gamma r)]\,e^{-inz}\,dn. \tag{17}$$

Putting $\gamma^2 = s^2+\frac{1}{4}\beta^2$ in (17), we get

$$p = \frac{-iK\,e^{i\omega t}}{\pi}\int_0^{\infty}[J_0\{\lambda r(s^2+\tfrac{1}{4}\beta^2)^{\frac{1}{2}}\}-iY_0\{\lambda r(s^2+\tfrac{1}{4}\beta^2)^{\frac{1}{2}}\}]\cos(zs)\,ds. \tag{18}$$

The integral in (18) is evaluated using the results given in Erdélyi *et al.* (1954, vol. 1, pp. 55, 56) and the expressions for p coincide with the expressions given in (9) and (10).

Further, if we solve the problem as an initial-value problem with an oscillatory source maintained for all $t > 0$, we obtain, in the limit $t \to \infty$, the solution satisfying the radiation condition directly.

The author wishes to express his gratitude to Professor Sir James Lighthill for going through the paper and making many suggestions for its improvement.

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